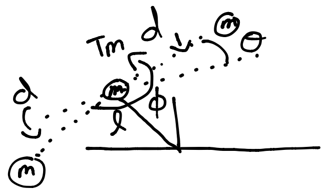
Apply Lagrange's Equations with generalized Forces to develop a model for an idealized tight-rope walker



Tm : torque applied by tight rope walker to the balance beam

generalized coordinates of tight-rope walker

→ ϕ : lean angle relative to vertical

→ θ : rotation angle of the balance beam relative to walkers hips

$L = T - V$ T: total KE, V: total PE

$\frac{d}{dt}\frac{\partial L}{\partial \dot\phi} - \frac{\partial L}{\partial \phi} = 0$ $\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} - \frac{\partial L}{\partial \theta} = 0$ if $T_m = 0$

If $T_m \neq 0$, a non-conservative force $\frac{d}{dt}\frac{\partial L}{\partial \dot\phi} - \frac{\partial L}{\partial \phi} = Q_\phi$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} - \frac{\partial L}{\partial \theta} = Q_\theta$$

$\begin{bmatrix} Q_\phi \\ Q_\theta \end{bmatrix}$ → generalized force vector   Corresponding to Tm   → generalized coordinate space $\begin{bmatrix} \phi \\ \theta \end{bmatrix}$

determine Q_ϕ by determining virtual work done by virtual displacement $d\phi$

how much work happens when we tweak ϕ and freeze θ

definition: $\delta W_\phi = Q_\phi \delta\phi = 0$ in this case

therefore: $Q_\phi = \frac{0}{d\phi} = 0$

determine Q_θ using virtual work

$\delta W_\theta = Q_\theta f_\theta = T_m \, \delta\theta$ so $Q_\theta = T_m$ (Lagrange = Tm)

Get Cartesian position coordinates of each mass as functions of ϕ, θ

differentiate to get the velocities, then $T = \frac{1}{2}mv^2$, $V = mgh$

$X_m = -l\sin\phi$ $\quad$ $Y_m = l\cos\phi$ $\quad$ $\dot{X}_m = -l\dot{\phi}\cos\theta$ $\quad$ $\dot{Y}_m = -l\dot{\phi}\sin\phi$

$V_a = mgY_m = mgl\cos\phi$ $\quad$ $T_a = \frac{1}{2}m(\dot{x}_m^2 + \dot{y}_m^2) = \frac{1}{2}ml^2\dot{\phi}^2$

now for right most mass M:

$X_{mr} = -l\sin\phi + d\cos(\phi+\theta)$ $\quad$ $Y_{mr} = -l\cos\theta + d\sin(\phi+\theta)$

$\dot{X}_{mr} = -l\dot{\phi}\cos\phi - d(\dot{\theta}+\dot{\phi})\sin(\theta+\phi)$ $\quad$ $\dot{Y}_{mr} = -l\dot{\phi}\sin\phi + d(\dot{\phi}+\dot{\theta})\cos(\phi+\theta)$

$V_b = MgY_{mr} = Mg[l\cos\phi + d\sin(\phi+\theta)]$

$T_b = \frac{1}{2}M(\dot{x}_{mr}^2 + \dot{y}_{mr}^2) = \frac{1}{2}M[l^2\dot{\phi}^2 + ld\dot{\phi}(\dot{\phi}+\dot{\theta})\sin\theta + d^2(\dot{\phi}+\dot{\theta})^2]$

Left-most mass: d is opposite

$V_c = Mg[l\cos\phi - d\sin(\phi+\theta)]$

$T_c = \frac{1}{2}M[l^2\dot{\phi}^2 - ld\dot{\phi}(\dot{\phi}+\dot{\theta})\sin\theta + d^2(\dot{\phi}+\dot{\theta})^2$

Get total KE's, total PE's

$$T_{tot} = T_a + T_b + T_c = \frac{1}{2}(m+2M)\,\ell^2\dot\phi^2 + Md^2(\dot\phi+\dot\theta)^2$$

$$V_{tot} = V_a + V_b + V_c = (m+2M)\,\ell g\cos\phi$$

$$\underbrace{\frac{d}{dt}\frac{\partial T}{\partial\dot\phi}}_{\frac{\partial L}{\partial\dot\phi}} \underbrace{- \frac{\partial T}{\partial\phi} + \frac{\partial V}{\partial\phi}}_{-\frac{dL}{\partial\phi}} = 0$$

V does not depend on $\dot\phi$

$$\frac{d}{dt}\left((m+2M)\,\ell^2\dot\phi + 2Md^2(\dot\phi+\dot\theta)\right) - [m+2M]\,\ell g\cos\phi = 0$$

$$\left[(m+2M)\,\ell^2 + 2Md^2\right]\ddot\phi + 2Md^2\ddot\theta - [m+2M]\,\ell g\sin\phi = 0 \quad (1)$$

$$\frac{d}{dt}\left|\frac{\partial T}{\partial\dot\theta}\right| - \frac{dT}{\partial\theta} + \frac{\partial V}{\partial\theta} = T_m \longrightarrow \frac{d}{dt}\left[2Md^2(\dot\phi+\dot\theta)\right] = T_m$$

$$[2Md^2]\,\ddot\phi + [2Md^2]\,\ddot\theta = T_m \quad (2)$$

solve for $\ddot\phi$ and $\ddot\theta$

$$\ddot\phi = \frac{g}{\ell}\sin\theta - \left[\frac{1}{(m+2M)\ell^2}\right]T_m \qquad \ddot\theta = -\frac{g}{\ell}\sin\phi + \left[\frac{m\ell^2 + 2M(\ell^2+d^2)}{2Md^2(m+2M)\,\ell^2}\right]T_m$$

does $\dot E = T_m\dot\theta$?

$$\dot E = \frac{dT}{d\dot\phi}\ddot\phi + \frac{\partial T}{\partial\dot\theta}\ddot\theta + \frac{\partial V}{\partial\phi}\dot\phi$$

$$\frac{dT}{\partial \dot{\phi}}\left[\frac{g}{l}\sin\theta - (\ldots)T_m\right] + \frac{dT}{\partial \dot{\theta}}\left[-\frac{g}{l}\sin\phi + (\ldots)T_m \quad + \frac{\partial V}{\partial \phi}\dot{\phi} \quad \ldots = T_m\dot{\theta}\right.$$

if you grab the balance beam: $\theta = 0$, $\dot{\phi} = 0$, $\ddot{\theta} = 0$

$0 = -\frac{g}{l}\sin\phi + [\ldots]\,T_m$ solve for T_m and put in $\ddot{\phi}$ equation

$$\ddot{\phi} = \left[\frac{(m+2M)l^2}{(m+2M)l^2 + 2Md^2}\right]\frac{g}{l}\sin\phi$$